**Lecture #20**

**A. BIRTH & DEATH PROCESS**

**Definition** Let { X(t) } be a discrete random process which represents the size of a certain population at time t. Two types of random events cause a change in the value of X(t) over time: a *birth* causes X(t) to increase by 1, and a *death* causes it to decrease by 1.

The discrete random process { X(t) } is called a *birth and death process* if the following postulates are satisfied:

Let n = X(t).

1. Prob[ 1 birth in (t,t+Dt) ] = nlDt + O(Dt2)

2. Prob[ 0 birth in (t,t+Dt) ] = 1 - nlDt + O(Dt2)

3. Prob[ 2 or more births in (t,t+Dt) ] = O(Dt2)

4. Prob[ 1 death in (t,t+Dt) ] = nmDt + O(Dt2)

5. Prob[ 0 death in (t,t+Dt) ] = 1 - nmDt + O(Dt2)

6. Prob[ 2 or more deaths in (t,t+Dt) ] = O(Dt2)

7. X(t) is independent of the number of occurrences of these two events in any time interval before or after (0,t).

8. The probability that these two events occur a certain number of times in time interval (t0,t0+t) depends on t, but not on t0.

NOTE: The birth and death rates, respectively l and m, are per unit of population and independent of time.

Derivation of the probability law:

Let Pn(t) = Prob[ X(t) = n ].

From the postulates, we can deduce WTMD that:

Pn’(t) = (n-1)lPn-1(t) – n(l+m)Pn(t) + (n+1)mPn+1(t) [Eqn. 1]

🡪 How? Think about the three ways in which change or no change X(t) = n can come about in the time interval between t and t+Dt. Group the terms, divide by Dt, and take the limit as Dt goes to zero.

A very similar but simpler equation applies when n = 0. We shall omit the details. The initial population n0 must be assumed to be > 0.

**B. SINGLE-SERVER POISSON QUEUE**

In a queue, such as at a bank counter, customers *arrive* , usually *wait* (unless there are no waiting customers), and receive *service* some time later. A very similar system operates inside a router or server.

(1) We assume that the number of arrivals and services are *independent of the number of customers present in the system*.

[Note that this differs from what we have assumed above for a birth and death process.]

(2) Also, we shall study the queue when it is in *steady state* – which means that the probabilities Pk are not changing with time; Pk’ = 0.

(3) The service rate has to be zero when n = 0.

If we simplify Eqn. 1 keeping in mind these points, and drop the time variable t, we get the following two key equations of the queue:

lPn-1 – (l+m)Pn + mPn+1 = 0

mP1 – lP0 = 0

Justification (simplified):

Assume the present state is n. Then, in the next time interval:

1. State n remains unchanged if (a) no arrival or service takes place, and (b) one arrival and one service takes place.

2. An arrival causes transition to state n+1.

3. Completed service causes state transition to state n-1, unless the queue is empty in which case there is no question of service.

Let r = l/m, sometimes called the *load factor*. Then, from the second equation, we get P1 = rP0. This can be plugged into the first equation, taking n = 2, 3 ... to give P2 = r2P0 ,P3 = r3P0 ...and so on.

But we know that the sum of all the Pk's must be 1. Thus we can solve for P0, and thereby all the Pk's. The solution is:

Pk = (1-r)rk for k = 0, 1, 2 ....

(1) The average number of customers in the system = S kPk , which can be shown to be equal to r/(1-r). Note that this increases without limit as the load factor r approaches 1.

(2) The probability density function (pdf) of the continuous RV wait time is exponential, with parameter (m-l). That is, f(w) = (m-l)e-(m-l)w.

[Recall the similar RV inter-arrival time in a Poisson process.]

From this, it follows that the expected waiting time is 1/(m-l).

Simple numerical example: single bank counter. See Excel sheet.